Intersection of Circles - Equal Areas Problem

Paul Sherard and Shidong Kan
University of Hawaii, Honolulu Community College
October 6, 2013

I. Problem

The original problem is described as: A farmer has a cow in a circular fenced-in pen. The farmer only wants the cow to eat one-half of the grass in the circular pen. He attempts this with a leash on the cow that is attached to the circular fence. The only question is: How long should the leash be relative to the radius of the pen?

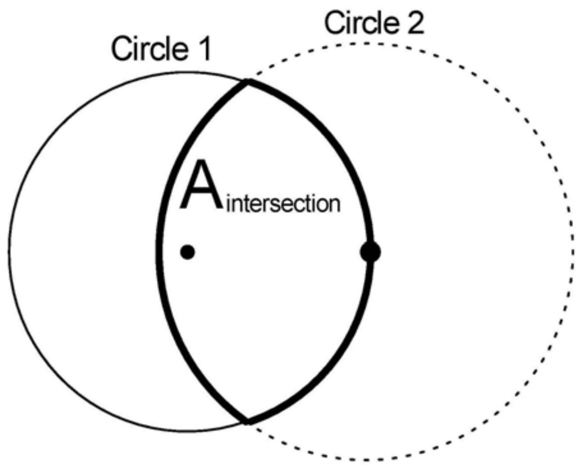

Fig. 1

Fig. 1 shows the geometry of the problem: With the center of Circle 2 on the perimeter of a Circle 1, determine the radius of the Circle 2 such that the area of the intersection of the two circles is one-half of Circle 1. In Fig. 1 this implies the condition: $A_{intersection}$ equals one-half the area of Circle 1. Thus, our "equal area condition" is:

$$A_{intersection} = (1/2)(\text{Area of Circle 1}) \qquad (1)$$

II. Geometric Solution

To solve the problem, the geometry is selected as shown in Fig. 2 below.

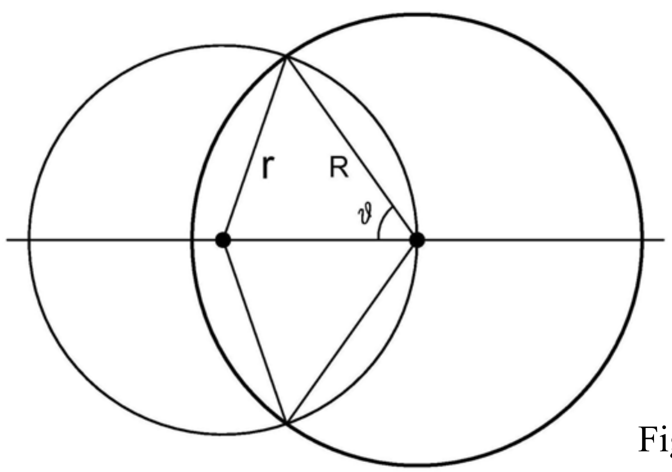

Fig. 2

The Circle 1 has a radius r and Circle 2 has a radius R. We have also defined the angle θ, as shown. We are trying to find the ratio R/r. It is obvious that for the equal area condition $R > r$. Thus, the ratio R/r will be greater than 1. To solve, the angle θ will be determined such that it satisfies the Eq. 1 equal-areas condition. To do this we must determine the area of the intersection of the two circles, $A_{intersection}$.

First let's get a relationship between R, r, and θ. We can draw the following triangle (Fig. 3):

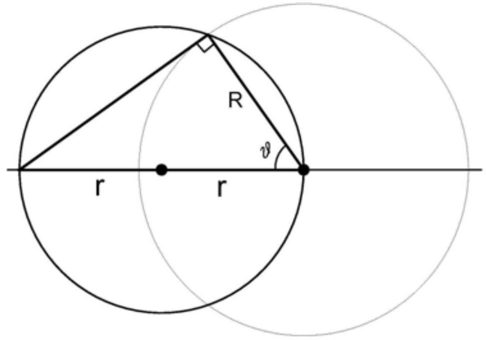

Fig. 3

We see that:

$$\cos\theta = \frac{R}{2r} \qquad (2)$$

Thus:

$$\boxed{\frac{R}{r} = 2\cos\theta} \qquad (3)$$

Now will attempt to find θ such that the equal-areas condition holds. We will do this by determining the area of the intersection. First consider the area of A_1, as shown in Fig. 4:

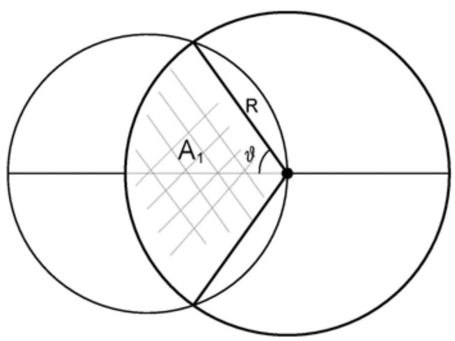

Fig. 4

We see that:

$$A_1 = \left(\frac{2\theta}{2\pi}\right)\pi R^2 = \theta \cdot R^2 \qquad (4)$$

Now we will determine the area of the remaining portion of the intersection area. This is the wedge section (A_4) shown in Fig. 5. By symmetry, the total contribution of the area is $2\times A_4$.

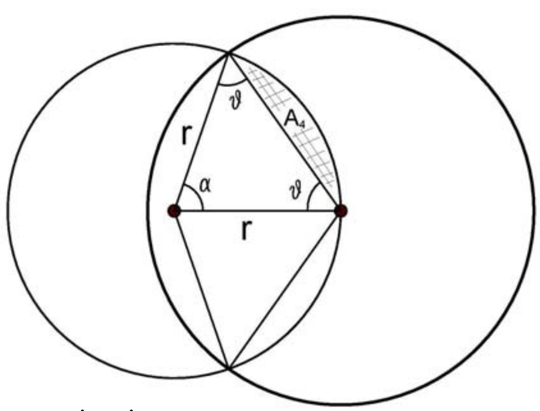

Fig.5

Therefore area of the intersection is:

$$A_{intersection} = A_1 + 2A_4 \qquad (5)$$

Now we will determine the area A_4 show in Fig. 5.

First consider the area of the triangle shown in Fig. 6.

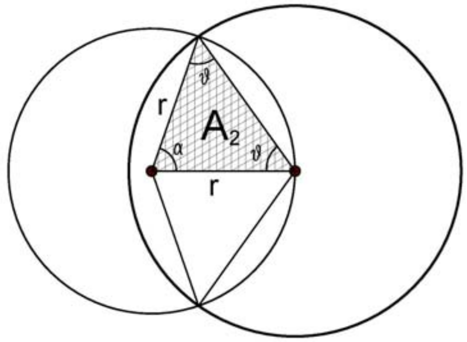

Fig. 6

$$A_2 = \frac{1}{2} r \cdot r \sin(\alpha) = \frac{1}{2} r^2 \sin(\alpha) \qquad (6)$$

Where $\alpha = \pi - 2\theta$, thus:

$$A_2 = \frac{1}{2} r^2 \sin(\pi - 2\theta) \qquad (7)$$

$$A_2 = \frac{1}{2} r^2 \sin(2\theta) \qquad (8)$$

Now consider area of the entire wedge A_3, as shown in Fig. 7:

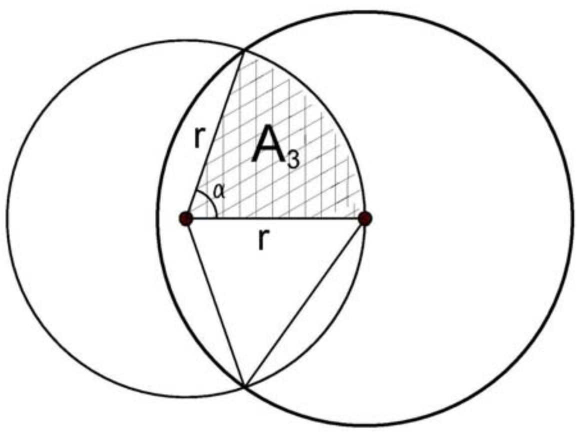

Fig. 7

Where:

$$A_3 = \left(\frac{\alpha}{2\pi}\right)\pi r^2 = \frac{\alpha}{2} r^2 \qquad (9)$$

Thus:

$$A_3 = \frac{1}{2}(\pi - 2\theta) r^2 \qquad (10)$$

Now we determine the area A_4 of the remaining portion of the intersection as shown in Fig. 5.

Whereby $A_4 = A_3 - A_2$, thus:

$$A_4 = \frac{1}{2}(\pi - 2\theta) r^2 - \frac{1}{2} r^2 \sin(2\theta) = \frac{1}{2} r^2 \left[(\pi - 2\theta) - \sin(2\theta)\right] \qquad (11)$$

From Eq. 5, $A_{intersection} = A_1 + 2A_4$, thus:

$$A_{intersection} = \theta \cdot R^2 + r^2[(\pi - 2\theta) - \sin(2\theta)] \tag{12}$$

[See **Appendix 2** for a check of the limits to this equation.]

Our condition is that $A_{intersection} = (1/2)\pi r^2$, thus:

$$\frac{1}{2}\pi r^2 = \theta \cdot R^2 + r^2\left[(\pi - 2\theta) - \sin(2\theta)\right]$$

We divide by r^2:

$$\frac{\pi}{2} = \theta\left(\frac{R}{r}\right)^2 + \pi - 2\theta - \sin(2\theta) \tag{13}$$

From Eq. 3:

$$\left(\frac{R}{r}\right)^2 = 4\cos^2\theta \tag{14}$$

Finally, we substitute Eq. 14 into Eq. 13. This gives an equation in terms of θ only. We then set everything equal to zero:

$$\boxed{4\theta\left(\cos^2\theta\right) - \sin(2\theta) - 2\theta + \frac{\pi}{2} = 0} \tag{15}$$

To solve problem, we must find the root of this equation (for $\theta < 90°$). Once we determine the solution for θ, then we can use Eq. 3, to solve for $R/r \rightarrow R/r = 2\cos(\theta)$.

Since we cannot solve Eq, 15 analytically, we must resort to solving for θ numerically

We find that: $\theta \cong 0.952848$ rad, thus, $R/r = 2\cos(\theta)$:

$$\boxed{\frac{R}{r} \approx 1.15873} \tag{16}$$

[See **Appendix 1** for detailed discussion of numerical analysis.]

Appendix 1: Numerical Solution

We plot the functions:

$$f(\theta) = 4\theta(\cos^2\theta) - \sin(2\theta) - 2\theta + \frac{\pi}{2}$$

$$g(\theta) = 0$$

The intersection of these curves yields the roots of Eq. 15. The plots shown in Figs. A1-1 and A1-2, below, are for $f(\theta)$ and $g(\theta)$ at two different scales.

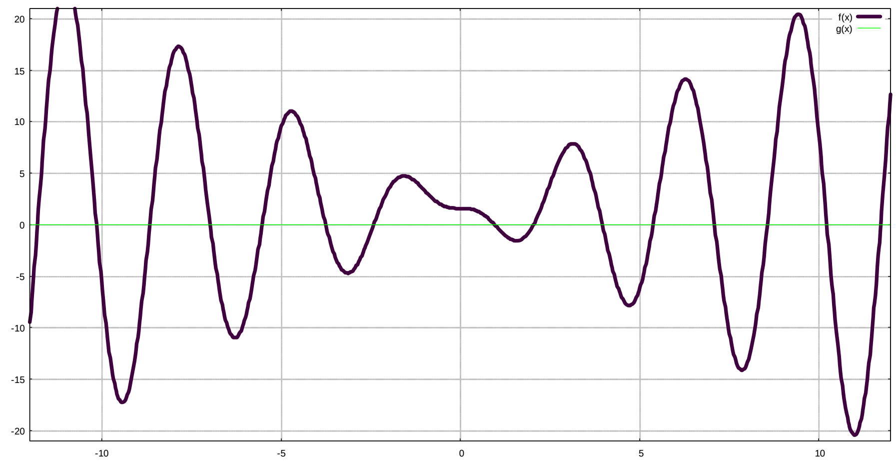

Fig. A1-1

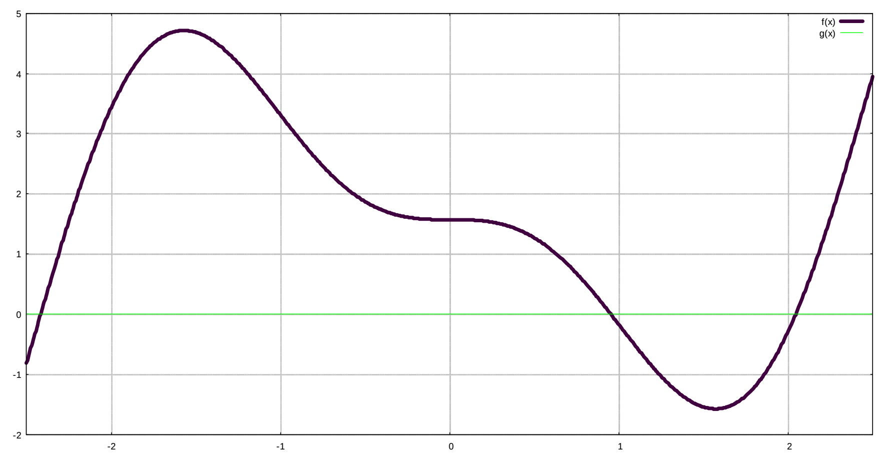

Fig. A1-2

We know that the solution is in the range $\theta < \pi/2$ ($\theta < 1.571$).

Thus, we find that:
$\theta \cong 0.952848$ rad ($54.594°$) [$f(\theta) \sim 10^{-9}$], thus $R/r = 2\cos(\theta) \cong 1.15873$

$$\theta = 0.952848 \text{ rad } (54.594°)$$

$$\boxed{\frac{R}{r} \approx 1.15873}$$ Solution.

Appendix 2: Check of Limits to the Solution for $A_{intersection}$

From Eq. 12:

$$A_{intersection} = \theta \cdot R^2 + r^2[(\pi - 2\theta) - \sin(2\theta)]$$

We will consider two limiting cases for this solution.

1. No intersection

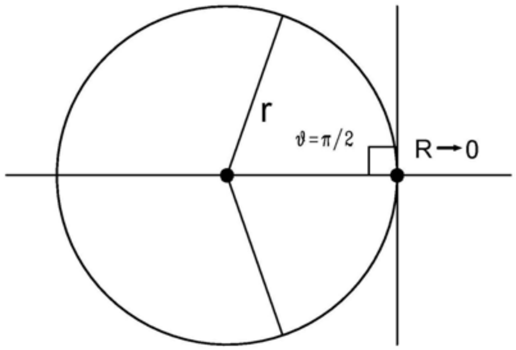

Fig. A2-1

In this limiting case, as shown in Fig. A2-1, the radius R of circle two goes to zero.
For $R \to 0$, thus $\theta \to \pi/2$, we expect $\boldsymbol{A_{intersection} = 0}$
Therefore: $2\theta = 2(\pi/2) = \pi$; $\sin(2\theta) = \sin(\pi) = 0$.

From Eq. (12) we have:
$A_{intersection} = (\pi/2)(0) + r^2[(\pi - \pi) - 0]$
$A_{intersection} = 0 + r^2[0 - 0] = 0$ (as expected!).

2. Full intersection − Circle 2 encloses Circle 1.

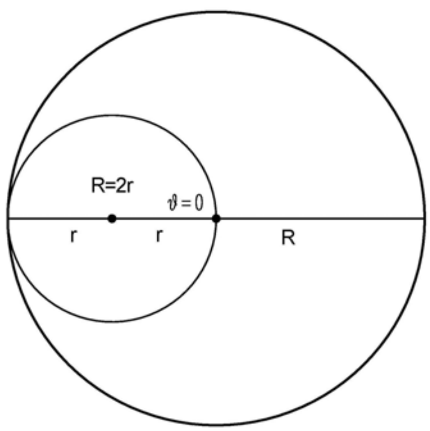

Fig. A2-2

In this limiting case, as shown in Fig. A2-2, the radius R of Circle 2 is increased to entirely enclose Circle 1. The intersection in this limiting case is simply the area of Circle 1.

For $R = 2r$, thus $\theta \to 0$, we expect $\boldsymbol{A_{intersection} = \pi r^2}$.
Therefore: $2\theta = 0$, $\sin(2\theta) = \sin(0) = 0$

From Eq. (12) we have:
$A_{intersection} = 0(2r)^2 + r^2[(\pi - 0) - 0] = 0 + r^2[\pi]$
$A_{intersection} = \pi r^2$ (as expected!)